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Benjamin–Ono interacting solitons as field representatives of Galilean point particles

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Abstract. The isomorphism between the algebras of conserved integrals for Benjamin-Ono interacting solitons and Galilean point particles has been proved.

1. Introduction

We consider two Hamiltonian dynamical systems in one space dimension. The first one corresponds to the finite-dimensional case of N Galilean particles with unit masses, described by the following equation of motion:

$$u_t = \bar{\theta}\nabla H \qquad H = \tfrac{1}{2}\sum_i p_i^2 \tag{1}$$

where $u = (q_1, \ldots, q_N, p_1, \ldots, p_N)^{\mathrm{T}}$, $\bar{\theta} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ is a standard implectic operator leading to the following Poisson bracket:

$$\{f, g\}_{\bar{\theta}} = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right) \tag{2}$$

and ∇H denotes the gradient of the Hamiltonian H. The second one corresponds to the infinite-dimensional system made of N solitons described by the Benjamin-Ono (BO) equation of motion [1, 2]

$$u_t = K_1 = 4auu_x + \mathscr{H}u_{xx} = \theta\nabla H \qquad H = \int_{-\infty}^{\infty} (\tfrac{1}{2}u\mathscr{H}u_x + \tfrac{2}{3}au^3)\,\mathrm{d}x \tag{3}$$

where $u = u(x, t)$ is the field function, $\mathscr{H}$ denotes the Hilbert transform, $\theta = D$ is an implectic operator leading to the following Poisson bracket:

$$\{F, G\}_\theta = \int_{-\infty}^{\infty} (\nabla F)D(\nabla G)\,\mathrm{d}x \qquad \nabla F = E(f) \qquad F = \int_{-\infty}^{\infty} f\,\mathrm{d}x. \tag{4}$$

Here D stands for the total x derivative and E is the Euler operator. The system (3) belongs to the class of systems completely integrable by the inverse scattering method [3].

Our aim is to show that N Galilean point particles and N interacting BO solitons $u^{(i)}$ [4] have isomorphic algebras of conserved integrals. This equivalence of both dynamic systems leads to the identification of each interacting soliton $u^{(i)}$ as a field representative of the ith point particle.

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The algebras of constants of motion are built up using the idea of master integrals [5, 6]. Let M be a Poisson manifold and $\mathscr{L}$ a Poisson algebra with respect to the bracket $\{\ ,\ \}$. Then, let $\mathscr{L}'$ be the Abelian subalgebra spanned by the integrals of motion H_i of some dynamical system and let $\hat{H}$ denote the map $\hat{H}: F \to \{H, F\}$ on $\mathscr{L}$. $T_m \in \mathscr{L}$ is called a master integral of degree m for this system if, for an arbitrary $H_1, \ldots, H_m \in \mathscr{L}'$, $\hat{H}_1 \hat{H}_2 \ldots \hat{H}_m T_m \in \mathscr{L}'$. Moreover, we have $\{T_m, T_n\} = T_{m+n-1}$, so the master integrals constitute an algebra. Now, if $H \in \mathscr{L}'$ and T_m is a master integral of degree m then [5]

$$\Pi_m(t) = \sum_{k=1}^{m} \frac{t^k}{k!} (\hat{H})^k T_m \tag{5}$$

is a time-dependent constant of motion for the Hamiltonian system with the Hamiltonian H, i.e.

$$\frac{\mathrm{d}\Pi}{\mathrm{d}t} = \frac{\partial \Pi}{\partial t} + \{H, \Pi\} = 0. \tag{6}$$

In this language, the time-independent constants of motion H_i are master integrals of degree 0. For a finite, integrable system with n degrees of freedom, we have n functionally independent master integrals of arbitrary degree. Moreover, master integrals of degree 0 and 1 are non-canonical action-angle variables [6] of this system as $\mathrm{d}T_0/\mathrm{d}t = 0$ and $\mathrm{d}T_1/\mathrm{d}t = \text{constant}$.

Remark. From the definitions of master integral and conserved integral (5), we find that the map $\mathrm{e}^{t\hat{H}}: T_m \to \Pi_m$ is the isomorphism for two Poisson subalgebras. Hence there is no difference whether we investigate directly conserved quantities or master integrals.

Let S be a tangent bundle of a phase space and $\mathscr{L}_s$ be a Lie algebra of vector fields with respect to the Lie bracket $[\ ,\ ]$. For a given evolution equation, we may introduce a concept of master symmetries τ and time-dependent symmetries $\sigma(t)$ in analogy to the concept of master integrals and time-dependent conserved integrals [5]. Of course, the Noether map $\theta\nabla$ is the Lie isomorphism between the algebra of master integrals and the algebra of Hamiltonian master symmetries of the considered dynamical system.

In § 2 of this paper we construct the algebra of master integrals for Galilean particles, in § 3 we do the same for the BO equation and in the last section we relate both results for the soliton surface of the BO system.

2. An algebra of master integrals and conserved functions for Galilean particles

Constants of the motion for non-interacting particles, i.e. master integrals of degree 0, are

$$\bar{H}_n \equiv \bar{T}_{0,n} = \frac{1}{n+1} \sum_i p_i^{n+1} \qquad n = -1, 0, 1, \ldots. \tag{7}$$

Note that $\bar{H}_{-1}$ is the number of particles N, $\bar{H}_0$ is the total momentum P and $\bar{H}_1$ is the total energy of the system. It is known [5] that, if we know the simplest master integral of degree 2 for (7), then we can generate all other master integrals by simple commutation. We found, in our case, the simplest master integral of degree 2 in the form $\bar{T}_{2,-1} = \Sigma_i\, q_i^2$. Hence all master integrals can be generated recursively:

$$\{\bar{T}_{k,n}, \bar{T}_{2,-1}\}_{\bar{\theta}} = 2^{n-k} \bar{T}_{k+1,n-1}. \tag{8}$$

The particular choice of structure constants will be clear when we consider the BO solitons. According to (7) and (8) we find

$$\bar{T}_{k,n} = \bar{c}(k, n) \sum_i q_i^k p_i^{n+1} \qquad \bar{c}(k, n) = (-1)^k \frac{(n+k)!}{(n+1)!} \frac{1}{2^{nk}}. \tag{9}$$

Hence, for example: $\bar{T}_{1,-1} = -2 \sum_i q_i$, $\bar{T}_{1,0} = -\sum_i q_i p_i$, $\bar{T}_{1,1} = -\frac{1}{2} \sum_i q_i p_i^2$. The algebra of master integrals (9) takes the form

$$\{\bar{T}_{k,n}, \bar{T}_{l,m}\}_{\bar{\theta}} = \frac{\bar{c}(k, n)\bar{c}(l, m)}{\bar{c}(k+l-1, n+m)} [k(m+1) - l(n+1)] \bar{T}_{k+l-1,m+n} \tag{10}$$

so its subalgebra of non-canonical action-angle variables is as follows:

$$\{\bar{T}_{0,n}, \bar{T}_{0,m}\}_{\bar{\theta}} = 0 \qquad \{\bar{T}_{0,n}, \bar{T}_{1,m}\}_{\bar{\theta}} = \frac{n+m+1}{2^m} \bar{T}_{0,n+m}$$

$$\{\bar{T}_{1,n}, \bar{T}_{1,m}\}_{\bar{\theta}} = (n-m) \bar{T}_{1,n+m}. \tag{11}$$

For Galilean particles with Hamiltonian $\bar{H}_1$ we find

$$\hat{H}_1^m \bar{T}_{k,n} = \bar{c}(k, n) \frac{k!}{(k-m)!} \sum_i q_i^{k-m} p_i^{n+m+1}. \tag{12}$$

Thus time-dependent conserved functions, fulfilling (6), are of the form

$$\bar{\Pi}_{k,n}(t) = \sum_{m=0}^{k} \frac{t^m}{m!} \hat{H}_1^m \bar{T}_{k,n} = \bar{c}(k, n) \sum_i p_i^{n+1}(q_i - p_i t)^k. \tag{13}$$

We notice that all $\bar{\Pi}_{k,n}(t)$ are expressible through the canonical action-angle variables (q_i, p_i).

3. An algebra of Hamiltonian master symmetries and related algebra of master integrals for the BO equation

The algebra of master integrals for the BO equation (3) can be constructed from the algebra of master symmetries via the inverse Noether map. The simplest master symmetries of degree 1 are [7]

$$\tau_{1,-1} = \frac{1}{2a} \qquad \tau_{1,0} = xu_x + u \qquad \tau_{1,1} = x(4auu_x + \mathscr{H}u_{xx}) + 2au^2 + \tfrac{3}{2}\mathscr{H}u_x. \tag{14}$$

Applying $\tau_{1,1}$, Fokas and Fuchssteiner [7] generated the whole hierarchy of symmetries K_n for the BO equation. In their notation these three master symmetries for $a = \frac{1}{2}$ were denoted by τ_0, τ_1 and τ.

Now, we would like to estimate the values of the appropriate structure constants in the following commutation relations:

$$[K_n, \tau_{1,-1}] = c(n)K_{n-1} \qquad [K_n, \tau_{1,0}] = d(n)K_n \qquad [K_n, \tau_{1,1}] = f(n)K_{n+1}. \tag{15}$$

Observation 1

$$d(n) = n+1 \tag{16a}$$

$$2(n+1) = f(n)c(n+1) - f(n-1)c(n). \tag{16b}$$

Proof. The relation $[\tau_{1,1}, \tau_{1,0}] = \tau_{1,1}$ and the Jacobi identity imply

$$\begin{aligned} f(n)[K_{n+1}, \tau_{1,0}] &= [[K_n, \tau_{1,1}], \tau_{1,0}] \\ &= -[[\tau_{1,0}, K_n], \tau_{1,1}] - [[\tau_{1,1}, \tau_{1,0}], K_n] \\ &= d(n)[K_n, \tau_{1,1}] + [K_n, \tau_{1,1}] \\ &= f(n)(d(n)+1)K_{n+1}. \end{aligned} \tag{17}$$

Hence, $d(n+1) = d(n)+1$ and from $d(0) = 1$ we find (16a). Then, analogously, applying the Jacobi identity to $[K_n, \tau_{1,0}] = 2[K_n, [\tau_{1,1}, \tau_{1,-1}]]$ we obtain (16b).

So, we have some arbitrariness in choosing $c(n)$ and $f(n)$. In [7], for example, $f(n) = 1$ and $c(n) = n(n+1)$, but we make another choice. We define K_n in such a way that the one-soliton solution of the flow $u_t = K_n$ has the velocity being the nth power of the one-soliton velocity of the BO flow. This is realised by choosing $c(n) = 2n$ and $f(n) = \frac{1}{2}n + 1$, and the first few K_n are

$$\begin{aligned} K_0 &= u_x \\ K_1 &= (2au^2 + \mathcal{H}u_x)_x \\ K_2 &= (\tfrac{16}{3}a^2u^3 + 4a\mathcal{H}uu_x + 4au\mathcal{H}u_x - \tfrac{4}{3}u_{xx})_x \\ K_3 &= (16a^3u^4 + 16a^2u^2\mathcal{H}u_x + 16a^2\mathcal{H}u^2u_x + 16a^2u\mathcal{H}uu_x \\ &\qquad + 4a(\mathcal{H}u_x)^2 + 4a\mathcal{H}u\mathcal{H}u_x - 2\mathcal{H}u_{xxx} - 12auu_{xx} - 8au_x^2)_x \\ &\vdots \end{aligned}$$

We shall come back to this choice in the next section.

As the simplest master symmetry of degree 2 is $\tau_{2,-1} = x/2a$, we may define an arbitrary $\tau_{k,n}$ through the relation

$$[\tau_{k,n}, \tau_{2,-1}] = 2^{n-k}\tau_{k+1,n-1} \tag{18}$$

which is consistent with our choice of $K_n = \tau_{0,n}$, $\tau_{1,-1}$, $\tau_{1,0}$ and $\tau_{1,1}$. Another useful relation has the form

$$[\tau_{k,n}, \tau_{1,-1}] = g(k, n)\tau_{k,n-1}. \tag{19}$$

Observation 2. The structure constant $g(k, n)$ has the value $(n+k)/2^{k-1}$.

Proof. Expressing $\tau_{k,n}$ from (19) by $\tau_{k-1,n+1}$ and $\tau_{2,-1}$ according to (18) and applying the Jacobi identity we find $g(k-1, n+1) = 2g(k, n)$. As $g(0, n) = 2n$, we obtain the result from observation 2.

Now, we come to the algebra of master integrals for the BO equation. First, one should notice that all $\tau_{k,n}$ are Hamiltonian master symmetries. It follows directly from (18) and from the fact that both $K_n = \tau_{0,n}$ and $\tau_{2,-1}$ are Hamiltonian master symmetries as well. Then, applying the inverse Noether map to (19) and (18), we find

$$T_{k,n-1} = \frac{1}{4a}\frac{2^k}{n+k}\int_{-\infty}^{\infty} \gamma_{k,n}\,\mathrm{d}x \tag{20}$$

and

$$\int_{-\infty}^{\infty} \gamma_{k,n}\,\mathrm{d}x = \frac{n+k}{2^n}\int_{-\infty}^{\infty} x\gamma_{k-1,n}\,\mathrm{d}x \tag{21}$$

where $\tau_{k,n} = D\gamma_{k,n} = D\nabla T_{k,n}$. Moreover, applying (21) recursively, we may define densities of master integrals through the adjoint symmetries $D^{-1}K_n = \gamma_{0,n}$ and the powers of x variable:

$$T_{k,n} = c(k,n)\int_{-\infty}^{\infty} x^k \gamma_{0,n+1}\,\mathrm{d}x \qquad c(k,n) = \frac{1}{4a}\frac{(n+k)!}{(n+1)!}\frac{1}{2^{nk}}. \tag{22}$$

Let us find the structure of the Lie algebra of master integrals. The leading term of the adjoint symmetry $\gamma_{0,n}$ is of the form

$$L\gamma_{0,n} = r(n)u^{n+1} \qquad r(n) = (4a)^n \frac{n!}{(n+1)!}. \tag{23}$$

Assuming that the leading term uniquely determines each master integral we can easily find

$$\{T_{k,n}, T_{l,m}\}_\theta = \frac{c(k,n)c(l,m)}{c(k+l-1, n+m)} 4a[l(n+1) - k(m+1)]T_{k+l-1,n+m} \tag{24}$$

where in particular, for non-canonical action-angle variables,

$$\begin{aligned} \{T_{0,n}, T_{0,m}\}_\theta &= 0 \\ \{T_{0,n}, T_{1,m}\}_\theta &= \frac{n+m+1}{2^m} T_{0,n+m} \\ \{T_{1,n}, T_{1,m}\}_\theta &= (n-m)T_{1,n+m}. \end{aligned} \tag{25}$$

Due to the isomorphism of the algebra of master symmetries and master integrals, the structure constants from (24) may be calculated through the Lie bracket of master symmetries as well, but then the calculations are much more cumbersome.

4. Soliton surface

The aim of this section is to calculate the values of master integrals for the BO equation on a soliton surface, i.e. for $u = u_N$, where u_N means the N-soliton solution of (3). First, we notice that, although the BO equation does not possess the local hereditary recursion operator in one space dimension, similar to equations with such an operator, all flows $u_t = K_n$ are associated with the same linear problem and K_n constitutes an Abelian subalgebra of symmetries [8]. Hence the N-soliton solution for each flow has the same form, differing only in soliton velocities, as was shown in [9]. So, we are looking for such a decomposition $u_N = \Sigma_i u^{(i)}$ that

$$K_m[u_N] = \sum_i \lambda_i(m) u_x^{(i)} \tag{26}$$

and asymptotically, as $t \to \pm\infty$, $u^{(i)}$ tends to a single soliton $u_s^{(i)}$. For hierarchies with a recursion operator, K_n were determined uniquely with $\lambda_i(m)$ being the powers of asymptotic speeds of solitons [9, 10]. For the BO hierarchy we have chosen the structure constant $f(m)$ in such a way that $\lambda_i(m)$ are also the powers of soliton velocities p_i, i.e. $\lambda_i(m) = p_i^m$. Moreover, the decomposition (26) introduces a suitable basis

$$b_{ik} = \int_{-\infty}^{\infty} x^k u^{(i)}\,\mathrm{d}x \tag{27}$$

in the algebra of master integrals such that

$$T_{k,n}=c(k,n)\sum_i p_i^{n+1}b_{ik}. \tag{28}$$

Now, we would like to calculate the values of the basic master integrals (27), although we do not yet know an explicit form of $u^{(i)}$, but this is not necessary. From the property

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}\,b_{ik}=\text{constant} \tag{29}$$

which each basic master integral for an arbitrary moment of time has to fulfil, we may calculate (27) for $t\to\infty$ when $u^{(i)}\to u_s^{(i)}$. As

$$u_s^{(i)}=\frac{1}{a}\,\frac{p_i}{p_i^2(x+q_i(t))^2+1} \tag{30}$$

we have

$$\int_{-\infty}^{\infty}u^{(i)}\,\mathrm{d}x=\int_{-\infty}^{\infty}u_s^{(i)}\,\mathrm{d}x=\frac{\pi}{a}\qquad T_{0,n}=\frac{\pi}{4a^2}\,\frac{1}{n+1}\sum_i p_i^{n+1} \tag{31a}$$

$$\int_{-\infty}^{\infty}xu^{(i)}\,\mathrm{d}x=\int_{-\infty}^{\infty}xu_s^{(i)}\,\mathrm{d}x=-q_i\frac{\pi}{a}\qquad T_{1,n}=-\frac{\pi}{4a^2}\,\frac{1}{2^n}\sum_i q_ip_i^{n+1}. \tag{31b}$$

For $k>1$ the integrals (27) contain infinite terms. So let us remove the inconvenient infinities by defining the map $\rho:F\to\tilde{F}$ of the functional F into the value of its finite part. Hence we obtain

$$\tilde{b}_{ik}=(-1)^k\frac{\pi}{a}q_i^k\qquad \tilde{T}_{k,n}=(-1)^k\frac{\pi}{a}c(k,n)\sum_i q_i^kp_i^{n+1}=\frac{\pi}{4a^2}\bar{c}(k,n)\sum_i q_i^kp_i^{n+1}. \tag{32}$$

Comparing (9), (22) and (32), we find that, on the soliton surface, $\bar{\rho}=(4a^2/\pi)\rho$ is the bijective map of BO master integrals onto Galilean master integrals: $\bar{\rho}T_{k,n}=\bar{T}_{k,n}$. Moreover, according to (10) and (24), $\bar{\rho}$ is a Lie algebra isomorphism:

$$\bar{\rho}\{T_{k,n},T_{l,m}\}_\theta=\{\bar{\rho}T_{k,n},\bar{\rho}T_{l,m}\}_{\bar{\theta}}. \tag{33}$$

Finally, we would like to find the explicit form of interacting solitons $u^{(i)}$. The method of finding $u^{(i)}$ for hierarchies with a hereditary recursion operator was presented in [11].

Because the BO hierarchy K_n is an Abelian subalgebra, this method may be applied successfully and, as a result, one finds the interacting solitons obtained for the first time by Yoneyama [4]. The reader may find in [4] the proof of condition (29), i.e. that $u^{(i)}$ are conserved densities for which centres of gravity move at constant speeds. The explicit form of $u^{(i)}$ is: $D_{z_i}D^{-1}u_N(z_1,\ldots,z_N)$, where u_N is the N-soliton solution of the BO equation, $z_i=x+p_i+c_i$, D^{-1} is the inverse of D_x and D_{z_i} is the total z_i derivative.

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